CSCI 3210:
Computational Game Theory

Market Equilibria: Algorithmic Perspective

Ref: Ch 5 [AGT]

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1

Market

2

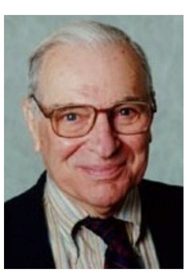
Fundamentals

- Competitive market
- Competitive equilibrium (CE)
- Partial equilibrium (PE)
- General equilibrium (GE)

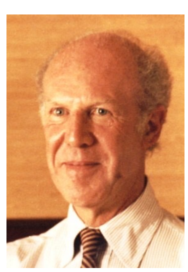
3

Literature

- Good news
  - CE **exists**
  - Proved by Arrow and Debreu (1954)
- Bad news
  - Existence proof is not algorithmic



Arrow



Debreu

6

Why is CE existence important?

- 1st Welfare Theorem
  - Any **CE** leads to a ***Pareto optimal*** allocation
  - Social significance

Nobody can be better off without making somebody worse off

- 2nd Welfare Theorem
  - Any Pareto opt outcome can be supported as a CE

7

Timeline

1954 – 2001

- We are happy. Equilibrium exists.
  Why bother about computation?
- Sporadic computational results
  - Eisenberg-Gail convex program, 1959
  - Scarf's computation of approximate fixed point, 1973
  - Nenakov-Primak convex program, 1983

8

Today’s markets

9

Electronic marketplaces

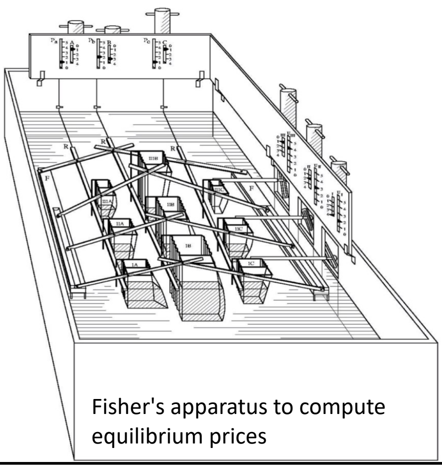
amazon ebay
AliExpress
Walmart

10

Fisher economy

Irving Fisher (1891)

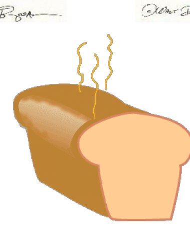
Mathematical model of a market

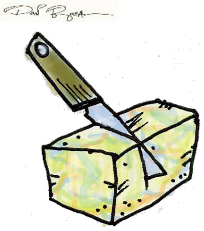


Fisher's apparatus to compute equilibrium prices

12

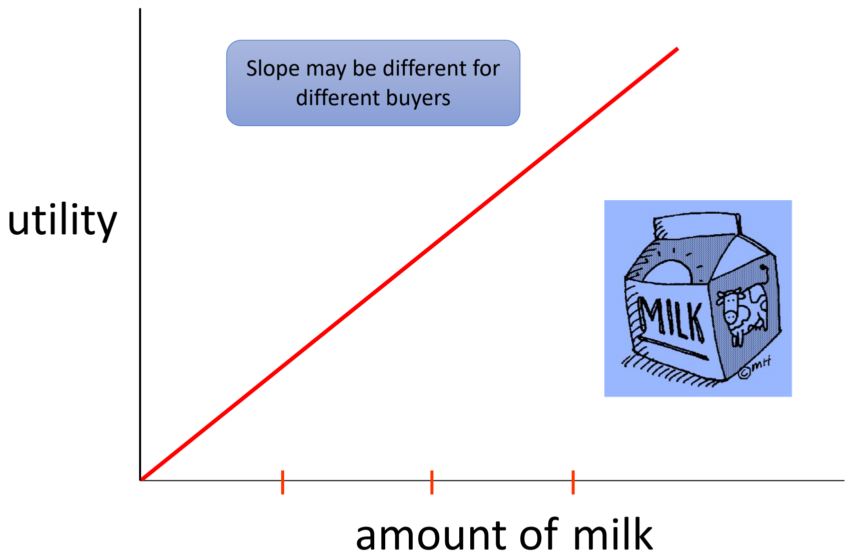
Fisher economy





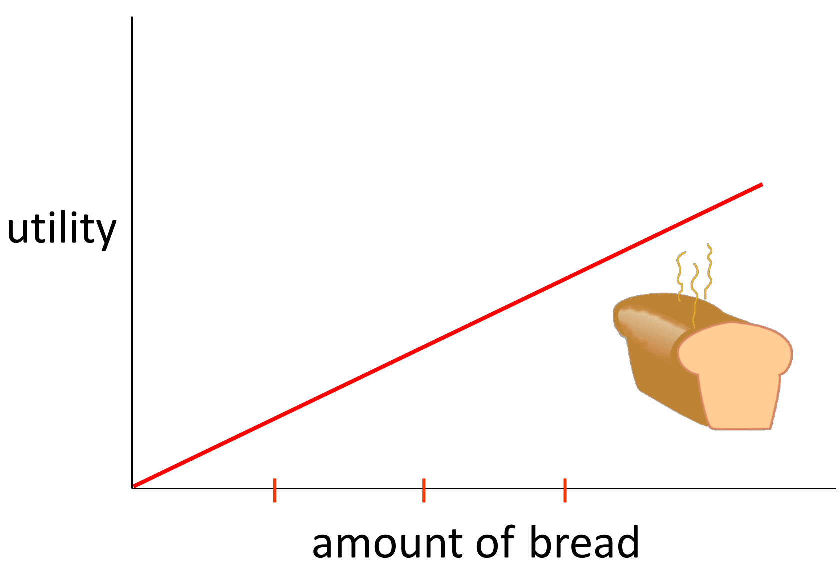
13

Utility function



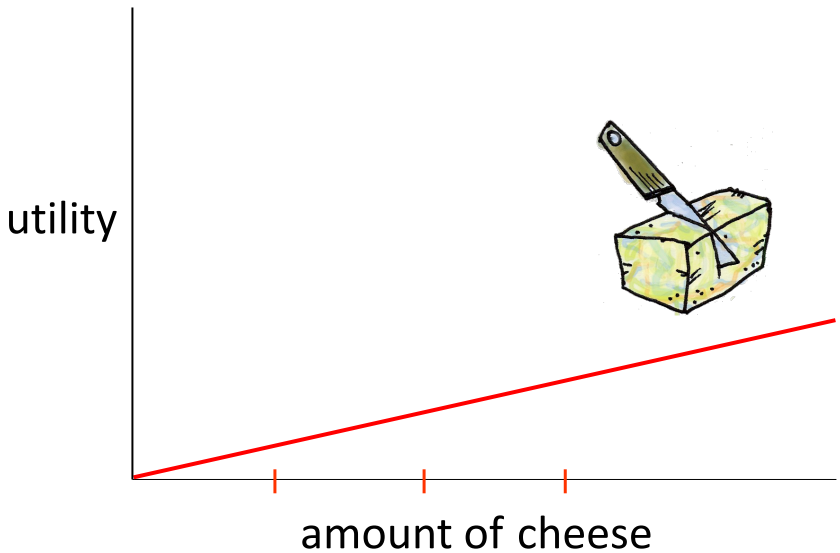
14

Utility function



15

Utility function



16

A buyer’s total utility

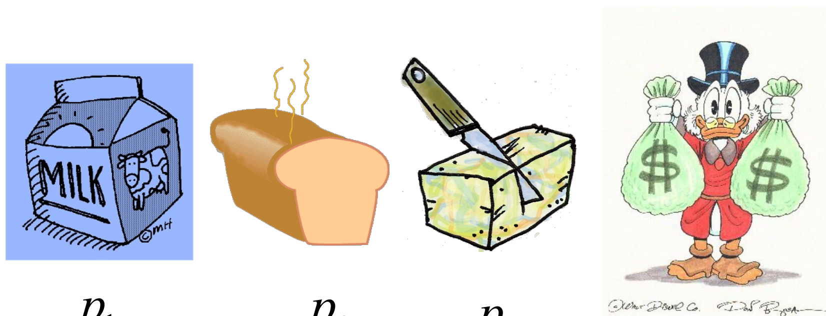
Total utility of a “bundle” of goods
= Sum of the utilities of individual goods

Optimal bundle of goods maximizes the total utility

17

Starter problem

- 1 unit of each good; one buyer
- Given: Prices, utility functions, and buyer's budget
- Give an algorithm to compute an **optimal** bundle for the buyer



p_1 p_2 p_3

18

Fisher market – setup

- Multiple goods, fixed amount of each good
- Multiple buyers, with individual budgets and utilities

19

Equilibrium/market-clearing allocations and prices

- Demand = supply
  - All goods sold out and all money spent
- Each buyer maximizes utility
  - Maximizes BPB

21

Equilibrium properties

- A buyer buys a good => maximizes BPB
- Price of a good > 0 => that good is sold out

22

Can we formulate an optimization routine?

- Does LP work?
- Anything else?

24

Can we formulate this as an LP?

Think about equilibrium allocations x_{ij}

$$\max \sum_i u_i(x) = \max \sum_i \sum_j u_{ij} x_{ij}$$

subject to

$$\forall j \quad \sum_i x_{ij} \le 1$$

$$\forall i, j \quad x_{ij} \ge 0$$

25

This LP doesn't work

- Multiplying all utilities of one buyer by 2 shouldn't change the solution
- But…

Maximize $u_1(x) + \sum_{i \neq 1} u_i(x)$ does not necessarily

maximize $2u_1(x) + \sum_{i \neq 1} u_i(x)$

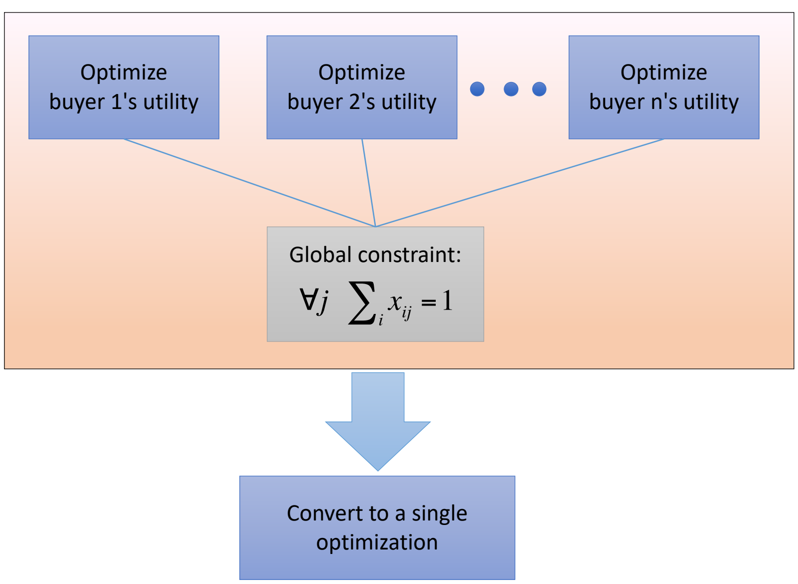
$$\forall j \quad \sum_i x_{ij} \le 1$$

$$\forall i, j \quad x_{ij} \ge 0$$

No LP formulation is known!

26

Main challenge



29

Review: Fisher model

- Model parameters (what's given)
  - ***n divisible goods*** (1 unit each wlog)
  - ***n' buyers***
  - e_i = buyer *i*'s budget (integral wlog)
  - u_{ij} = buyer *i*'s utility per unit of good *j* (int wlog)
    - *Linear* utility functions
- Want (not given):
  - x_{ij} = amount of good *j* that *i* buys to maximize utility $u_i(\mathbf{x}) = \sum_{j=1}^{n} u_{ij} x_{ij}$
  - p_j = price per unit of good j
  - No excess demand or supply

30

Eisenberg-Gale Formulation of Fisher Market

31

Eisenberg-Gale convex program

$$\max \sum_i e_i \log\left(\sum_j u_{ij}\, x_{ij}\right)$$

$$subject\ to$$

$$\sum_i x_{ij} \leq 1, \forall j$$

$$x_{ij} \geq 0, \forall i, j$$

Solving this gives market-clearing allocations (and prices as dual variables) iff every good gives a positive utility to some buyer.

Equilibrium prices are unique!

32

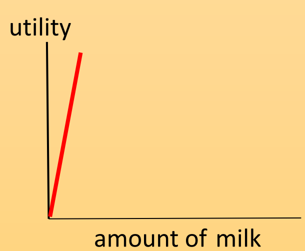
Example

2 buyers, 1 good (1 unit of milk)

Buyer 1

Budget, e_1 = \$100

u_{11} = 10/unit of milk

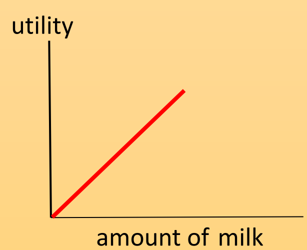


x^*_{11} = ?

Buyer 2

Budget, e_2 = \$50

u_{21} = 1/unit of milk



x^*_{21} = ?

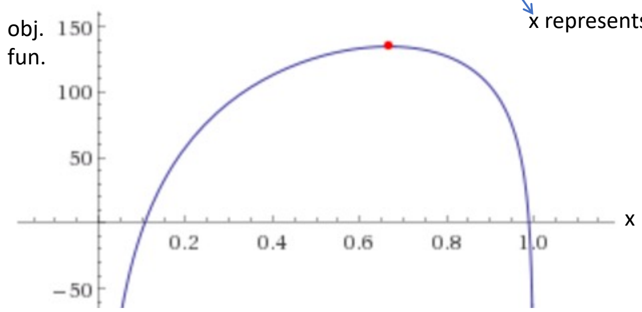
33

Solution

- $x_{11} = 2/3$, $x_{21} = 1/3$

maximize	function	$100 \log(10\,x) + 50 \log(1 - x)$
	domain	$0 \le x \le 1$

x represents x_{11}



34

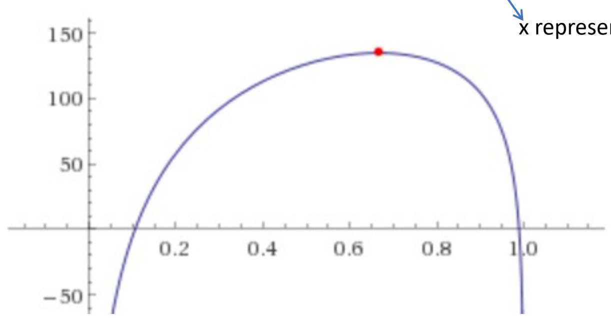
Solution

Why $x_{11} = 2/3$, $x_{21} = 1/3$?

Set price of milk = $150/unit

maximize	function	$100 \log(10\,x) + 50 \log(1 - x)$
	domain	$0 \le x \le 1$

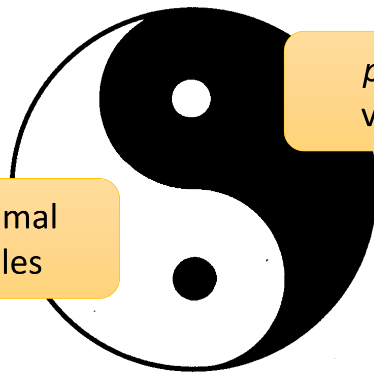
x represents x_{11}



35

Primal-dual

p_j
= The price of good j at an equilibrium
= Dual variable corresponding to the primal constraint for good j: $\sum_i x_{ij} \le 1$



36

How to solve nonlinear programs?

Lagrange function

KKT conditions

37

Eisenberg-Gale convex program

$$\max \sum_i e_i \log\left(\sum_j u_{ij}\, x_{ij}\right)$$
$$subject\ to$$
$$\sum_i x_{ij} \le 1, \forall j$$
$$x_{ij} \ge 0, \forall i, j$$

- Lagrange function

$$L(x,\lambda,\mu) = -\sum_i e_i \log \sum_j u_{ij}x_{ij} + \sum_j \lambda_j \left(\sum_i x_{ij} - 1\right) + \sum_i \sum_j \mu_{ij}(-x_{ij})$$

39

KKT conditions

- Stationary condition

$$\frac{e_i u_{ij}}{\sum_j u_{ij}\, x^*_{ij}} = \lambda^*_j - \mu^*_{ij} \qquad (1)$$

$$\frac{u_{ij}}{\lambda^*_j} \le \frac{\sum_j u_{ij} x^*_{ij}}{e_i}$$

- Primal feasibility

$$\sum_i x^*_{ij} \le 1, \forall j$$
$$x^*_{ij} \ge 0, \forall i, j$$

- Dual feasibility

$$\lambda^*_i, \mu^*_{ij} \ge 0, \forall i, j$$

- Complementary slackness

$$\lambda^*_j \left(\sum_i x^*_{ij} - 1\right) = 0 \quad \Leftrightarrow \quad \lambda^*_j > 0 \Rightarrow \sum_i x^*_{ij} = 1$$

$\mu^*_{ij}(-x^*_{ij}) = 0$ ⇔ $x^*_{ij} > 0$ => $\mu^*_{ij} = 0$

=> BPB = total utility/budget

40

Theorem: There exist market-clearing allocations (or prices) **iff** each good has an interested buyer (someone who gets positive utility for that good)

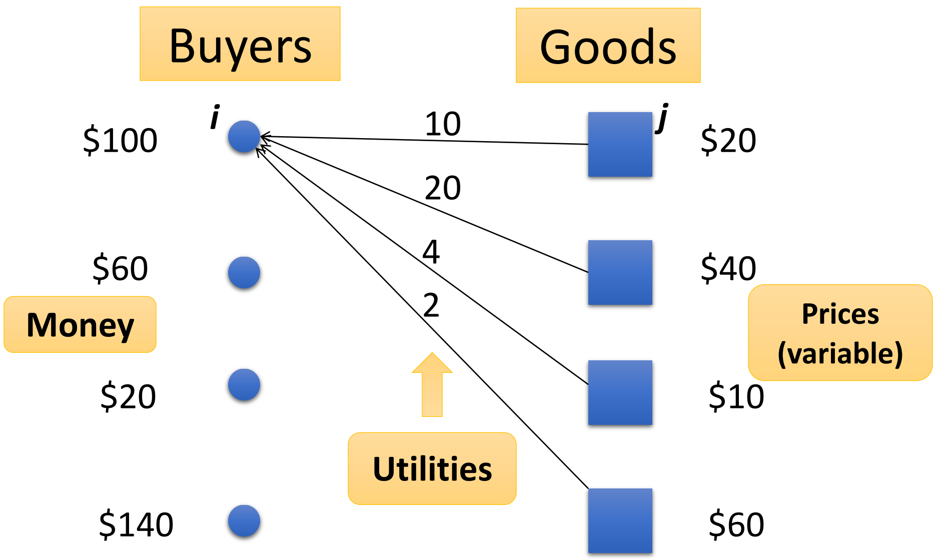
- Theorem 5.1 (AGT pg. 107)
- Prove that if each good has an interested buyer then
  1. All goods are sold out
  2. Each buyer spends all their money while maximizing their utility
- You prove the reverse direction

41

Network flow to solve Fisher market

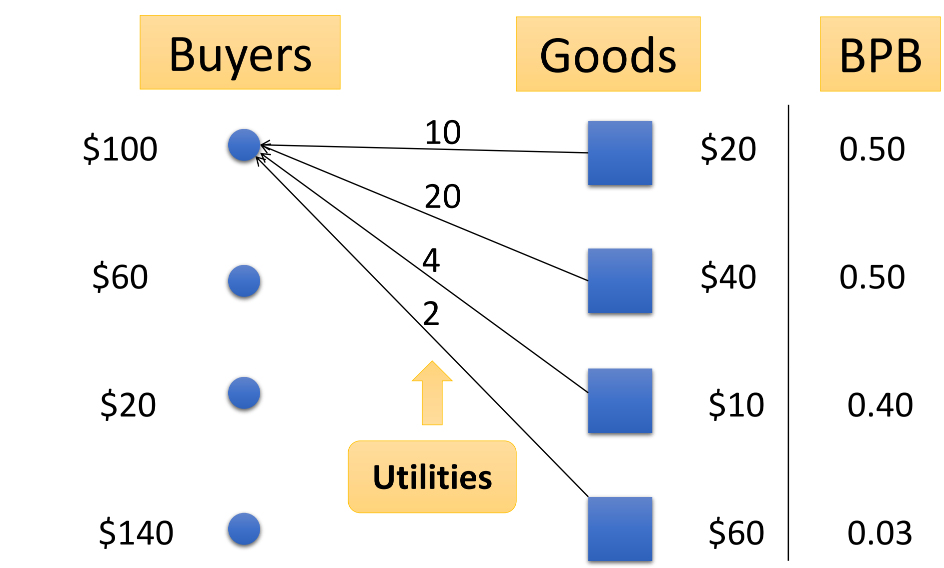
48

Example



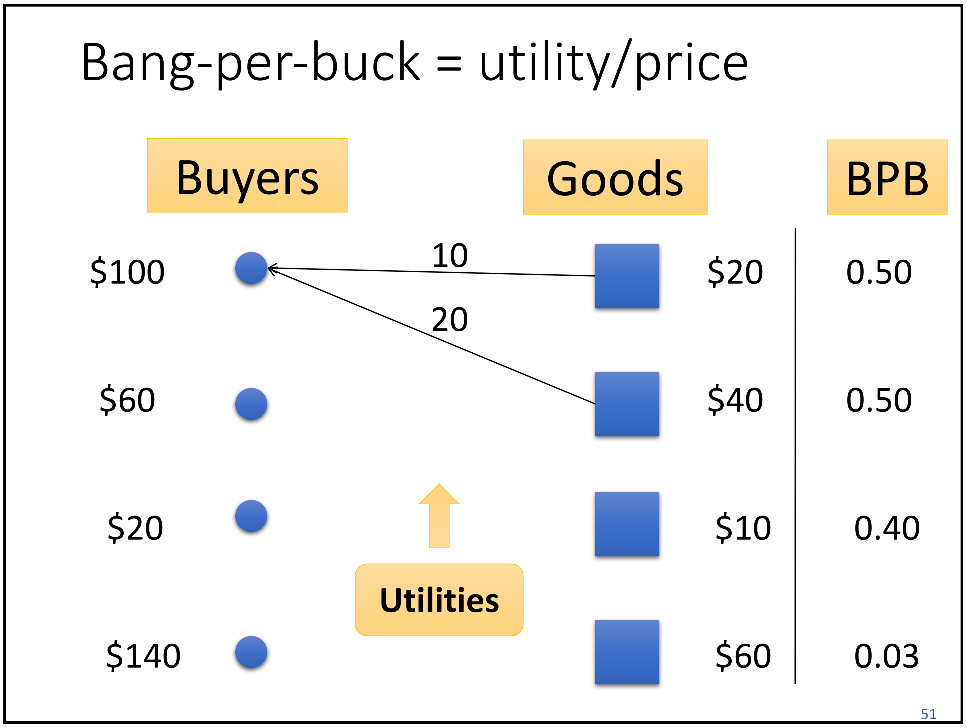
49

Bang-per-buck = utility/price

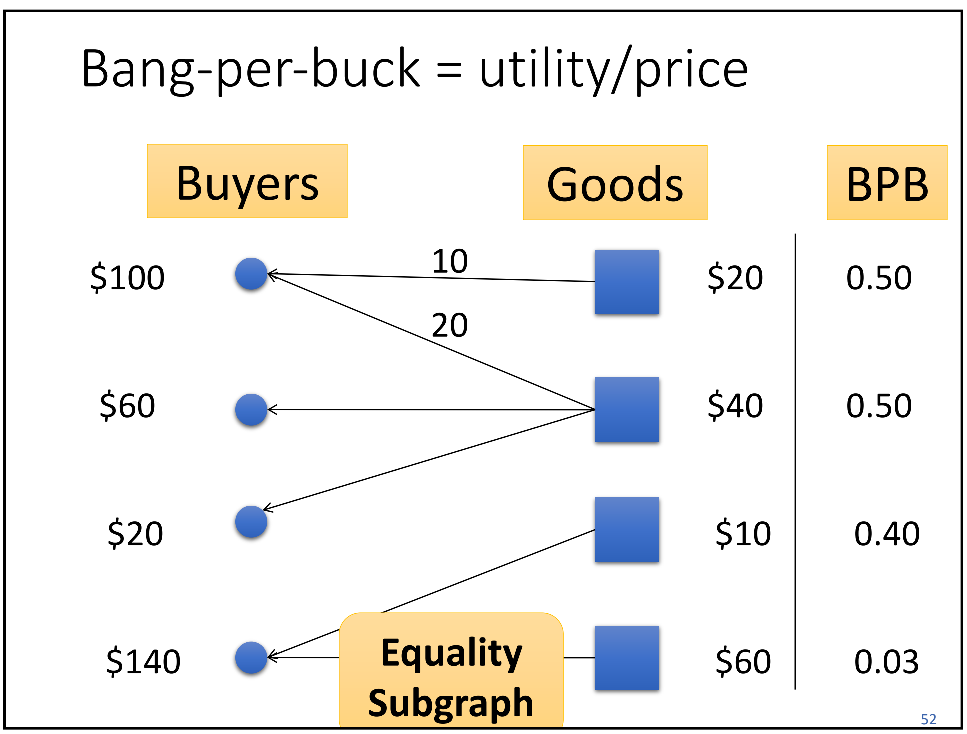


50

50


Bang-per-buck = utility/price
Buyers
Goods
BPB
$100
10
20
$20
0.50
$60
$40
0.50
$20
$10
0.40
Utilities
$140
$60
0.03
51

51


Bang-per-buck = utility/price
Buyers
Goods
BPB
$100
10
20
$20
0.50
$60
$40
0.50
$20
$10
0.40
$140
Equality Subgraph
$60
0.03
52

52

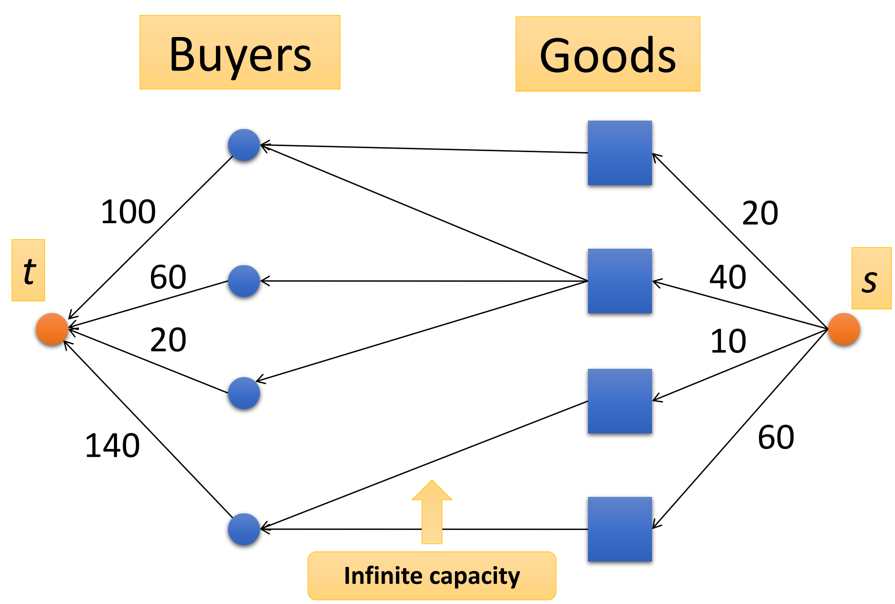
Equality subgraph

- Buyer is happiest when they can buy goods in equality subgraph
- How to maximize sales in the equality subgraph at a given price?

Use "max flow"!

53

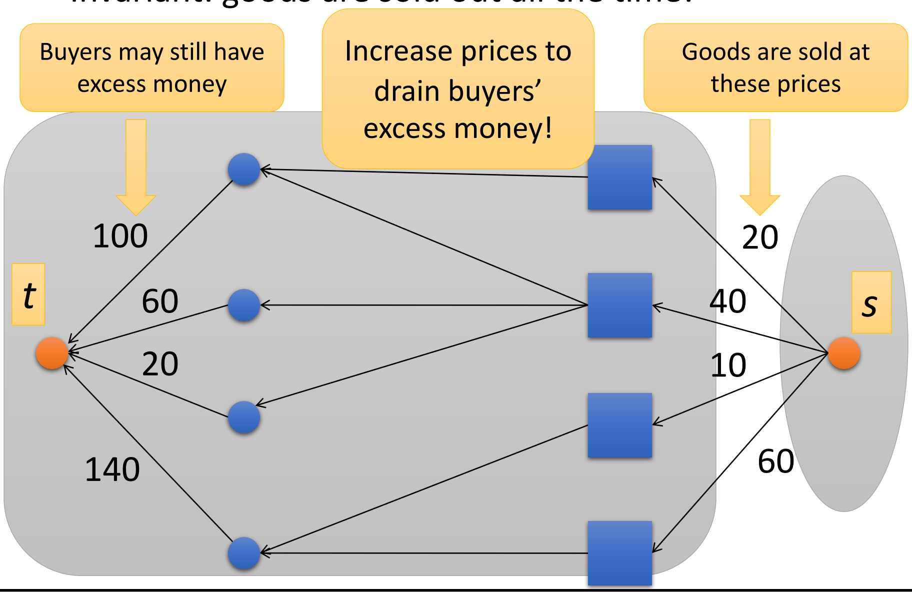
Max flow



54

Idea of the algorithm

Invariant: goods are sold out all the time!



55

Formal study of *network flow*

Next

56